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Abstract

For a massive scalar field with a general curvature coupling parameter, we
evaluate Wightman function, vacuum expectation values of the field square
and the energy–momentum tensor induced by a Z_2-symmetric brane with finite
thickness located on $(D + 1)$-dimensional AdS bulk. For the general case of the
static plane symmetric interior structure, the expectation values in the region
outside the brane are presented as the sum of free AdS and brane-induced
parts. For a conformally coupled massless scalar, the brane-induced part in the
vacuum energy–momentum tensor vanishes. In the limit of strong gravitational
fields, the brane-induced parts are exponentially suppressed for points not too
close to the brane boundary. As an application of general results, a special
model is considered in which the geometry inside the brane is a slice of the
Minkowski spacetime orbifolded along the direction perpendicular to the brane.
For this model the Wightman function, vacuum expectation values of the field
square and the energy–momentum tensor inside the brane are evaluated. It
is shown that for both minimally and conformally coupled scalar fields the
interior vacuum forces acting on the brane boundaries tend to decrease the
brane thickness.

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1. Introduction

Braneworlds naturally appear in string/M-theory context and provide a novel setting for
discussing phenomenological and cosmological issues related to extra dimensions. Motivated
by the problems of the radion stabilization and the generation of cosmological constant, the
role of quantum effects in braneworlds has attracted a great deal of attention [1–41]. A class
of higher dimensional models with compact internal spaces is considered in [42]. Many of
treatments of quantum fields in braneworlds deal mainly with the case of the idealized brane
with zero thickness. From a more realistic point of view, we expect that the branes have

a finite thickness. The finite core effects also lead to the modification of the Friedmann equation describing the cosmological evolution inside the brane. In string theory, there exists the minimum length scale and we cannot neglect the thickness of the corresponding branes at the string scale. The branes modeled by field theoretical domain walls have a characteristic thickness determined by the energy scale where the symmetry of the system is spontaneously broken. Various models are considered for a thick brane. Mainly, these models are constructed as solutions to the coupled Einstein-scalar equations by choosing a suitable potential for the scalar field. Vacuum fluctuations for a thick de Sitter brane supported by a bulk scalar field with an axion-like potential and the self-consistency of this braneworld are investigated in [36].

In the present paper based on [40], we describe the effects of core on properties of the quantum vacuum for a general plane symmetric static model of the brane with finite thickness. The most important quantities characterizing these properties are the vacuum expectation values of the field square and the energy–momentum tensor. Though the corresponding operators are local, due to the global nature of the vacuum, the vacuum expectation values describe the global properties of the bulk and carry an important information about the internal structure of the brane. As the first step for the investigation of vacuum densities, we evaluate the positive frequency Wightman function for a massive scalar field with a general curvature coupling parameter. This function gives comprehensive insight into vacuum fluctuations and determines the response of a particle detector of the Unruh–DeWitt type moving in the brane bulk. The problem under consideration is also of separate interest as an example with gravitational and boundary-induced polarizations of the vacuum, where all calculations can be performed in a closed form. The corresponding results specify the conditions under which we can ignore the details of the interior structure and approximate the effect of the brane by the idealized model. In addition, as it is shown below, the phenomenological parameters in the zero-thickness brane models such as brane mass terms for scalar fields are calculable in terms of the inner structure of the brane within the framework of the model considered in the present paper.

This paper is organized as follows. In section 2, we consider the Wightman function in the exterior of the brane for the general structure of the core with Poincare invariance along the directions parallel to the brane. By using the formula for the Wightman function, in section 3 we investigate the vacuum expectation values of the field square and the energy–momentum tensor. As an illustration of the general results, in section 4 we consider a model with Minkowskian geometry inside the brane. For this model, the vacuum expectation values inside the core are investigated as well. The last section contains a summary of the work.

2. Wightman function

We consider a brane with finite thickness $2a$ on the background of $(D + 1)$-dimensional AdS spacetime with the curvature radius $1/k_D$ (see figure 1). As in the Randall–Sundrum (RS) 1-brane scenario [43], we assume that the model is Z_2-symmetric with respect to the plane $y = 0$ located at the brane center. The spacetime is described by two distinct metric tensors in the regions outside and inside the brane. The corresponding line element has the form

$$ds^2 = \begin{cases} e^{-2k_D|y|}(dt^2 - d\mathbf{x}^2) - dy^2, & \text{if} \quad |y| > a, \\ e^{2u(y)}(dt^2 - d\mathbf{x}^2) - e^{2w(y)} dy^2, & \text{if} \quad |y| < a, \end{cases} \tag{1}$$

where $\mathbf{x} = (x^1, \ldots, x^{D-1})$ are the coordinates parallel to the brane. We assume that the geometry inside the brane is Poincaré invariant along these directions. Due to the Z_2-symmetry,
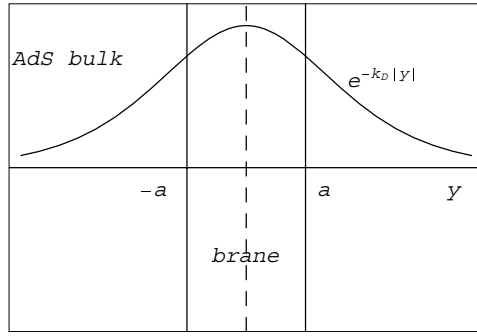
Figure 1. The geometry of a thick brane on AdS bulk.

the functions $u(y)$ and $w(y)$ are even functions of y. These functions are continuous at the core boundary: $u(a) = -k_D a$ and $w(a) = 0$. Here, we assume that an additional infinitely thin plane shell located at $|y| = a$ is present with the surface energy–momentum tensor τ_i^k, $\tau_D^D = 0$. Under the Israel matching conditions, one has

$$u'(a-) = -k_D + 8\pi G \tau_0^0 / (D-1), \qquad \tau_i^k = \tau_0^0 \delta_i^k, \qquad i = 1, 2, \ldots, D-1, \tag{2}$$

where G is the Newton gravitational constant.

We are interested in the vacuum polarization effects for a scalar field with the general curvature coupling parameter ξ propagating in the bulk described by line element (1). The corresponding field equation has the form

$$(\nabla_i \nabla^i + m^2 + \xi R)\varphi = 0, \tag{3}$$

where R is the Ricci scalar for the background spacetime. As a first stage for the evaluation of the vacuum expectation values (VEVs) for the field square and the energy–momentum tensor (EMT), we consider the positive frequency Wightman function. This function can be evaluated by using the mode sum formula

$$\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \sum_\alpha \varphi_\alpha(x) \varphi_\alpha^*(x'), \tag{4}$$

where $\{\varphi_\alpha(x), \varphi_\alpha^*(x')\}$ is a complete orthonormalized set of positive and negative frequency solutions to the field equation specified by the collective index α.

The eigenfunctions can be presented in the form

$$\varphi_\alpha(x^i) = \frac{e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}}{\sqrt{2\omega (2\pi)^{D-1}}} f_\lambda(y), \qquad \omega = \sqrt{k^2 + \lambda^2}, \qquad k = |\mathbf{k}|, \tag{5}$$

where λ is the separation constant. Below we will assume that $y \geqslant 0$. The corresponding formulae in the region $y < 0$ are obtained from the Z_2-symmetry of the model. Substituting eigenfunctions (5) into field equation (3), for the function $f_\lambda(y)$ one obtains the equation

$$e^{-Du-w} \partial_y [e^{Du-w} \partial_y f_\lambda] - (m^2 + \xi R - \lambda^2 e^{-2u}) f_\lambda = 0. \tag{6}$$

For the exterior AdS geometry one has $u(y) = -k_D y$, $R = -D(D+1)k_D^2$ and the solution to equation (6) is expressed in terms of cylinder functions. The solution in the region $y < a$ even in y we will denote by $R(y, \lambda)$, $R(-y, \lambda) = R(y, \lambda)$. The parameter λ enters in the radial equation in the form λ^2 and this solution can be chosen in such a way that $R(y, -\lambda) = \text{const} \cdot R(y, \lambda)$. Now for the eigenfunctions one has

$$f_\lambda(y) = \begin{cases} R(y, \lambda), & \text{if} \quad y < a, \\ e^{Dk_D y/2} [A_\nu J_\nu(\lambda z) + B_\nu Y_\nu(\lambda z)], & \text{if} \quad y > a, \end{cases} \tag{7}$$

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where A_ν and B_ν are the integration constants, $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel and Neumann functions, and we use the notations

$$\nu = \sqrt{D^2/4 - D(D+1)\xi + m^2/k_D^2}, \qquad z = \mathrm{e}^{k_D y}/k_D. \tag{8}$$

For a conformally coupled massless scalar $\xi = (D-1)/(4D)$, $\nu = 1/2$ and the cylinder functions in equation (7) are expressed in terms of elementary functions.

The radial function is continuous at $y = a$. In order to find the condition for its derivative, we note that the discontinuity of the function $u'(y)$ at $y = a$ leads to the delta function term $2D[u'(a-) + k_D]\delta(y-a)$ in the Ricci scalar and hence, in equation (6) for the radial eigenfunctions. For a non-minimally coupled scalar field, due to the delta function term in the equation for the radial eigenfunctions, these functions have a discontinuity in their slope at $y = a$. The corresponding jump condition is obtained by integrating the equation (6) through the point $y = a$

$$f_\lambda'(a+) - f_\lambda'(a-) = \frac{16\pi G D \xi}{D-1} \tau_0^0 f_\lambda(a). \tag{9}$$

Now the coefficients in the formulae (7) for the exterior eigenfunctions are determined by the continuity condition for the radial eigenfunctions and by the jump condition for their radial derivative. From these conditions for the radial part of the eigenfunctions in the region $y > a$ we find

$$f_\lambda(y) = \frac{\pi}{2}\, \mathrm{e}^{D k_D (y-a)/2} R(a,\lambda)[\bar{Y}_\nu(\lambda z_a) J_\nu(\lambda z) - \bar{J}_\nu(\lambda z_a) Y_\nu(\lambda z)]. \tag{10}$$

where $z_a = \mathrm{e}^{k_D a}/k_D$. Here and in what follows we use the notation

$$\bar{F}(z) \equiv z F'(z) + \left[\frac{D}{2} - \frac{16\pi G D \xi}{(D-1)k_D}\tau_0^0 - \frac{\partial_y R(y,\lambda)|_{y=a}}{k_D R(a,\lambda)} \right] F(z). \tag{11}$$

Note that due to our choice of the function $R(y,\lambda)$, the logarithmic derivative in formula (11) is an even function of λ. Under the orthonormalization condition for the radial eigenfunctions, one finds the relation

$$R^{-2}(a,\lambda) = \frac{\pi^2}{2} \frac{\bar{J}_\nu^2(\lambda z_a) + \bar{Y}_\nu^2(\lambda z_a)}{z_a^D k_D^{D-1} \lambda}, \tag{12}$$

which determines the normalization coefficient for the interior eigenfunctions.

Substituting the eigenfunctions (5) into the mode sum (4), under the condition $z + z' > 2z_a + |t - t'|$ the Wightman function can be presented in the form

$$\langle 0|\varphi(x)\varphi(x')|0 \rangle = \frac{1}{2}\langle 0_S|\varphi(x)\varphi(x')|0_S \rangle - \frac{k_D^{D-1}}{(2\pi)^D}(zz')^{\frac{D}{2}} \int \mathrm{d}\mathbf{k}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \int_k^\infty \mathrm{d}\lambda\, \lambda$$

$$\times \frac{\tilde{I}_\nu(\lambda z_a)}{\tilde{K}_\nu(\lambda z_a)} \frac{K_\nu(\lambda z) K_\nu(\lambda z')}{\sqrt{\lambda^2 - k^2}} \cosh\left[\sqrt{\lambda^2 - k^2}(t'-t)\right], \tag{13}$$

where $\langle 0_S|\varphi(x)\varphi(x')|0_S \rangle$ is the positive frequency Wightman function for the AdS spacetime without boundaries (see, for instance, [30]), and the second term on the right is induced by the brane. Here and below the tilted notation for the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ is defined by the formula

$$\tilde{F}(x) \equiv x F'(x) + \mathcal{R}(a,x) F(x), \tag{14}$$

with the notation

$$\mathcal{R}(a,x) = \frac{D}{2} - \frac{16\pi G D \xi}{(D-1)k_D}\tau_0^0 - \frac{\partial_y R(y, x\, \mathrm{e}^{\pi \mathrm{i}/2}/z_a)|_{y=a}}{k_D R(a, x\, \mathrm{e}^{\pi \mathrm{i}/2}/z_a)}. \tag{15}$$

4

Quantum effects in free AdS spacetime are well investigated in the literature (see references given in [30]) and in the discussion below we will be mainly concentrated on the effects induced by the brane.

As we see from (13), the information about the inner structure of the brane is contained in the logarithmic derivative of the interior radial function in formula (15). In the RS 1-brane model with the brane of zero thickness the brane-induced part in the Wightman function is given by a similar formula with the replacement [30]

$$\mathcal{R}(a, x) \rightarrow D/2 - 2D\xi - c/2k_D, \tag{16}$$

in the definition (14) of the tilted notation. The parameter c is the brane mass term for a scalar field which is a phenomenological parameter in the model with zero thickness brane. As we see, in the model under consideration the effective brane mass term is determined by the core structure. Note that in the RS 2-brane model the mass terms on the branes determine the 1-loop effective potential for the radion field and play an important role in the stabilization of the interbrane distance.

3. Vacuum expectation values outside the brane

Outside the brane, the local geometry is the same as that for the AdS spacetime, and the renormalization procedure for the local characteristics of the vacuum is the same as for the free AdS spacetime. By using the formula for the Wightman function from the previous section, the VEV of the field square in the exterior region is presented in the form

$$\langle 0|\varphi^2|0\rangle = \tfrac{1}{2}\langle 0_S|\varphi^2|0_S\rangle + \langle \varphi^2\rangle_b, \tag{17}$$

where $\langle 0_S|\varphi^2|0_S\rangle$ is the VEV of the field square in the free AdS spacetime. The part induced by the brane is obtained from the second term on the right of formula (13) in the coincidence limit:

$$\langle \varphi^2\rangle_b = -\frac{k_D^{D-1} z^D}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty \mathrm{d}x\, x^{D-1} \frac{\tilde{I}_\nu(xz_a)}{\bar{K}_\nu(xz_a)} K_\nu^2(xz). \tag{18}$$

The VEV of the field square in the free AdS spacetime is well investigated in literature [45] and does not depend on the spacetime point, which is a direct consequence of the maximal symmetry of the AdS bulk.

At large distances from the brane, $z \gg z_a$, we introduce a new integration variable $y = xz$ and expand the integrand over z_a/z. By using the formula for the integral involving the square of the MacDonald function, to the leading order we obtain

$$\langle \varphi^2\rangle_b = -\frac{k_D^{D-1}(z_a/z)^{2\nu}}{2^{D+2\nu+1}\pi^{(D-1)/2}} \frac{\mathcal{R}(a,0)+\nu}{\mathcal{R}(a,0)-\nu} \frac{\Gamma(D/2+\nu)\Gamma(D/2+2\nu)}{\nu\Gamma^2(\nu)\Gamma((D+1)/2+\nu)}. \tag{19}$$

As we see, at large distances from the brane the brane-induced part is exponentially suppressed by the factor $\exp(-2\nu k_D y)$.

Having the Wightman function and the VEV for the field square, the VEV of the EMT in the region $y > a$ can be evaluated by using the formula

$$\langle 0|T_{ik}|0\rangle = \lim_{x' \to x} \partial_i \partial_k' \langle 0|\varphi(x)\varphi(x')|0\rangle + \left[\left(\xi - \tfrac{1}{4}\right) g_{ik}\nabla_l\nabla^l - \xi\nabla_i\nabla_k - \xi R_{ik}\right]\langle 0|\varphi^2|0\rangle. \tag{20}$$

Note that on the left of this formula, we have used the expression for the EMT which differs from the standard one by the term which vanishes on the solutions of the field equation (3) (see [46]). Similar to the Wightman function, the components of the vacuum EMT are presented in the decomposed form

$$\langle 0|T_{ik}|0\rangle = \tfrac{1}{2}\langle 0_S|T_{ik}|0_S\rangle + \langle T_{ik}\rangle_b, \tag{21}$$

where $\langle 0_S | T_{ik} | 0_S \rangle$ is the vacuum EMT in the free AdS spacetime and the part $\langle T_{ik} \rangle_{\mathrm{b}}$ is induced by the brane. For a conformally coupled massless scalar field and for even values of D the renormalized free AdS part in the VEV of the EMT vanishes. For odd values of D, this part is completely determined by the trace anomaly (see [44]).

Substituting the expressions of the Wightman function and the VEV of the field square into formula (20), for the part of the EMT induced by the brane one obtains

$$\langle T_i^k \rangle_{\mathrm{b}} = -\frac{k_D^{D+1} z^D \delta_i^k}{(4\pi)^{D/2} \Gamma(D/2)} \int_0^\infty \mathrm{d}x \, x^{D-1} \frac{\tilde{I}_\nu(x z_a)}{\tilde{K}_\nu(x z_a)} F^{(i)}[K_\nu(xz)], \tag{22}$$

where for a given function $g(v)$ we have introduced the notations

$$F^{(i)}[g(v)] = \left(\frac{1}{2} - 2\xi\right) \left[v^2 g'^2(v) + \left(D + \frac{4\xi}{4\xi - 1}\right) v g(v) g'(v) \right. $$
$$\left. + \left(v^2 + v^2 + \frac{2v^2}{D(4\xi - 1)}\right) g^2(v) \right], \tag{23}$$

$$F^{(D)}[g(v)] = -\frac{v^2}{2} g'^2(v) + \frac{D}{2}(4\xi - 1) v g(v) g'(v)$$
$$+ \frac{1}{2}[v^2 + v^2 + 2\xi D(D+1) - D^2/2] g^2(v), \tag{24}$$

with $i = 0, 1, \ldots, D - 1$. For a conformally coupled massless scalar field one has $\nu = 1/2$ and from formulae (23) and (24) it follows that $F^{(i)}[K_\nu(x)] = F^{(D)}[K_\nu(x)] = 0$. Hence, in this case the brane-induced parts in the VEVs of the EMT vanish. Note that for a conformally coupled scalar and for even values of D the conformal anomaly is absent and the free AdS part in the vacuum EMT vanishes as well.

For large distances from the brane, $z \gg z_a$, introducing a new integration variable $y = xz$ we expand the integrand over z_a/z. To the leading order this leads to the result

$$\langle T_i^k \rangle_{\mathrm{b}} = -\frac{2^{1-D-2\nu} k_D^{D+1} \delta_i^k}{\pi^{D/2} \Gamma(D/2) \nu \Gamma^2(\nu)} \left(\frac{z_a}{z}\right)^{2\nu} \frac{\mathcal{R}(a) + \nu}{\mathcal{R}(a) - \nu} \int_0^\infty \mathrm{d}x \, x^{D+2\nu-1} F^{(i)}[K_\nu(x)]. \tag{25}$$

The integrals in this formula may be evaluated by using the formulae from [47]. Note that the free AdS parts in the VEVs of both field square and the EMT do not depend on the spacetime point and, hence, at large distances from the brane they dominate in the total VEVs. Noting that in the limit of strong gravitational field in the region outside the brane, corresponding to large values of k_D, one has $z/z_a = \mathrm{e}^{k_D(y-a)} \gg 1$, we see that formulae (19) and (25) also describe the asymptotic behavior of the brane-induced VEVs in this limit. Hence, in the limit of strong gravitational field, for the points not too close to the brane boundary, the brane-induced parts are exponentially suppressed. The free AdS parts behave as k_D^{D-1} and their contribution dominates for strong gravitational fields.

4. The model with flat spacetime inside the brane

4.1. The exterior region

As an application of the general results given above let us consider a simple example assuming that the spacetime inside the brane is flat. The corresponding models for the cosmic string and global monopole cores were considered in [48] and are known as flower-pot models. Under the continuity condition on the brane boundary, it follows that in coordinates (x^μ, y) for the interior functions one has $u(y) = -k_D a, w(y) = 0$. From the

matching condition (2) we find the corresponding surface EMT with the nonzero components $\tau_i^k = (D-1)k_D\delta_i^k/8\pi G$, $i = 0, 1, \ldots, D-1$. The corresponding surface energy density is positive. We consider the VEVs in the exterior and interior regions separately.

For the model under consideration, the interior radial eigenfunction with Z_2-symmetry, $R(-y, \lambda) = R(y, \lambda)$, has the form

$$R(y, \lambda) = \frac{2z_a^D k_D^{D-1}\lambda\cos(k_y y)}{\pi^2\cos^2(k_y a)\left[\bar{J}_v^2(\lambda z_a) + \bar{Y}_v^2(\lambda z_a)\right]}, \qquad k_y^2 = \lambda^2\, e^{2k_D a} - m^2, \tag{26}$$

where the normalization coefficient is found from formula (12) and the barred notation is defined by

$$\bar{F}(z) \equiv zF'(z) + [D/2 - 2\xi D + (k_y/k_D)\tan(k_y a)]F(z). \tag{27}$$

As a result, the parts in the Wightman function, in the VEVs of the field square and the EMT induced by the brane are given by formulae (13), (18) and (22) respectively, where the tilted notations for the modified Bessel functions are defined by (14) with the coefficient

$$\mathcal{R}(a, x) = D/2 - 2\xi D - \sqrt{x^2 + m^2/k_D^2}\tanh\left(ak_D\sqrt{x^2 + m^2/k_D^2}\right). \tag{28}$$

Comparing (28) with (16), we see that in the limit $a \to 0$ from the results of the model with flat interior spacetime the corresponding formulae in the RS 1-brane model with a zero thickness brane are obtained.

The VEVs of the field square and the EMT diverge on the boundary of the brane. The leading term in the corresponding asymptotic expansion for the field square is given by

$$\langle\varphi^2\rangle_b \approx \frac{k_D A_D}{2^{D+5}\pi^{(D+1)/2}}\frac{\Gamma((D-1)/2)}{(D-2)(y-a)^{D-2}}, \qquad D > 2, \tag{29}$$

with the notation

$$A_D = 4D - D^2 + 1 + 4\xi D(D-3) - 4m^2/k_D^2. \tag{30}$$

For $D \leqslant 2$, the VEV of the field square is finite on the core boundary. Note that in the model with zero thickness brane located at $y = 0$, the corresponding VEVs near the brane behave as y^{1-D}.

For the asymptotic behavior of the EMT we have (no summation over i)

$$\langle T_i^i\rangle_b \approx -\frac{k_D A_D(\xi - \xi_D)}{2^{D+4}\pi^{(D+1)/2}}\frac{\Gamma((D+1)/2)}{(y-a)^D}, \qquad \langle T_D^D\rangle_b \approx \frac{Dk_D(y-a)}{D-1}\langle T_0^0\rangle_b, \tag{31}$$

with $i = 0, 1, \ldots, D-1$. For a conformally coupled field (no summation over i)

$$\langle T_i^i\rangle_b \approx \frac{D-3}{Dk_D(y-a)}\langle T_D^D\rangle_b \approx \frac{m^2}{D}\langle\varphi^2\rangle_b, \qquad i = 0, 1, \ldots, D-1, \quad D > 3. \tag{32}$$

For $D = 3$ the radial stress $\langle T_D^D\rangle_b$ diverges logarithmically. In the case $D = 2$ the corresponding VEVs are finite on the boundary of the brane. In the limit $m/k_D \gg 1$ the brane-induced VEVs are suppressed by the factor $\exp[-2(m/k_D)\ln(z/z_a)]$.

On the left panel of figure 2, we have plotted the dependence of the brane-induced parts in the VEVs of the energy density and radial stress on z/z_a for a minimally coupled massless scalar field ($\xi = 0$) in the case $D = 4$. This parameter is related to the distance from the boundary of the brane by the formula $z/z_a = \exp[k_D(y-a)]$. Recall that for a conformally coupled massless scalar field the brane-induced VEVs vanish. Note that in $D = 4$ the conformal anomaly is absent and for massless scalar fields the VEV of the energy–momentum tensor in the free AdS spacetime is zero.
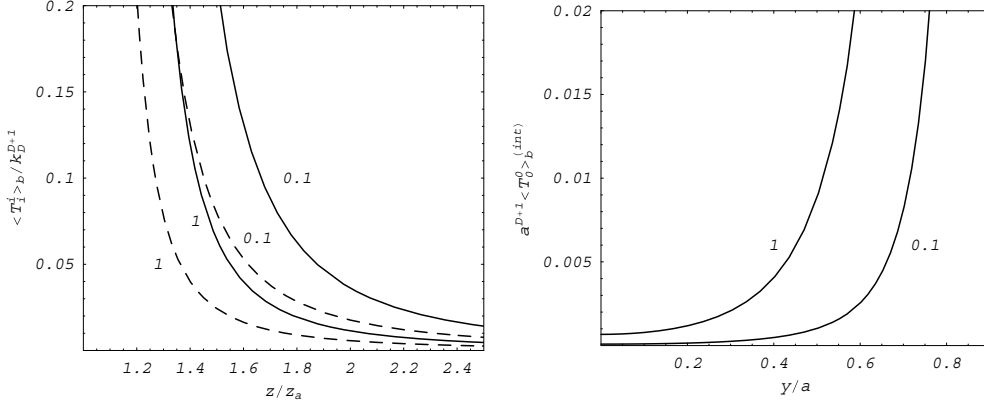
Figure 2. On the left panel, the part in the VEV of the energy density (full curves) and radial stress (dashed curves), $k_D^{-D-1}\langle T_i^i \rangle_{\rm b}$, $i = 0, D$, induced by the brane are plotted as the functions of z/z_a for a minimally coupled massless scalar field in $D = 4$. On the right panel, the corresponding energy density inside the brane, $a^{D+1}\langle T_0^0 \rangle_{\rm b}^{\rm (int)}$, induced by the AdS geometry in the exterior region is plotted as a function of y/a. The numbers near the curves correspond to the values of ak_D.

4.2. The interior region

Now let us consider the vacuum polarization effects inside the brane for the model with a flat interior. Substituting eigenfunctions (26) into the mode sum formula, the corresponding Wightman function is presented in the form

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = G_0(x, x') + G_1(x, x'),\tag{33}$$

where $G_0(x, x')$ is the Wightman function in the Minkowski spacetime orbifolded along the y-direction and

$$G_1(x, x') = -\frac{(z_a k_D)^D}{(2\pi)^D} \int d\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \int_k^\infty dx\, \frac{xC\{e^{-\varkappa(x)a}, K_\nu(xz_a)\}}{C\{\cosh(\varkappa(x)a), K_\nu(xz_a)\}}$$
$$\times \frac{\cosh(\varkappa(x)y)\cosh(\varkappa(x)y')}{\varkappa(x)\sqrt{x^2-k^2}} \cosh\left[\sqrt{x^2-k^2}(t-t')\right].\tag{34}$$

In (34), $\varkappa(x) = \sqrt{x^2 e^{2k_D a} + m^2}$, and we have used the notation

$$C\{f(u), g(v)\} = vf(u)g'(v) + [(D/2 - 2\xi D)f(u) - (u/ak_D)f'(u)]g(v),\tag{35}$$

with $g(v) = K_\nu(v)$ and $f(u) = e^{-u}$, $\cosh u$ for the numerator and denominator, respectively. The function $G_0(x, x')$ differs by the factor $1/2$ from the Wightman function for a plate in the Minkowski spacetime located at $y = 0$ on which the field obeys the Neumann boundary condition. The term $G_1(x, x')$ is induced by the AdS geometry in the region $y > a$. For a conformally coupled massless scalar field one has $\nu = 1/2$ and by using definition (35) it can be explicitly checked that $C\{e^{-u}, K_\nu(u/ak_D)\} = 0$. Hence, in this case the part $G_1(x, x')$ vanishes.

Now we turn to the evaluation of the renormalized VEV for the field square. The renormalization corresponds to the omission of the part coming from the Minkowskian Wightman function in (33). As a result, the VEV is presented in the form

$$\langle\varphi^2\rangle_{\rm ren}^{\rm (int)} = \langle\varphi^2\rangle_{0,\rm ren}^{\rm (int)} + \langle\varphi^2\rangle_{\rm b}^{\rm (int)}.\tag{36}$$

Here, the part $\langle\varphi^2\rangle^{(\text{int})}_{0,\text{ren}}$ is given by the formula

$$\langle\varphi^2\rangle^{(\text{int})}_{0,\text{ren}} = \frac{m^{D-1}}{2(2\pi)^{(D+1)/2}}\frac{K_{(D-1)/2}(2my)}{(2my)^{(D-1)/2}}, \tag{37}$$

and the second term is obtained from (34) in the coincidence limit,

$$\langle\varphi^2\rangle^{(\text{int})}_{\text{b}} = -\frac{(4\pi)^{-D/2}}{\Gamma(D/2)}\int_m^\infty \mathrm{d}x(x^2-m^2)^{D/2-1}\cosh^2(xy)U_\nu(x), \tag{38}$$

with the notation

$$U_\nu(x) = \frac{C\{\mathrm{e}^{-ax}, K_\nu(\sqrt{x^2-m^2}/k_D)\}}{C\{\cosh(ax), K_\nu(\sqrt{x^2-m^2}/k_D)\}}. \tag{39}$$

This part in the VEV of the field square is induced by the exterior AdS geometry.

The integral on the right of formula (38) is finite for $|y| < a$ and diverges on the boundary of the brane $|y| = a$. To the leading order, near the boundary $y = a$ we find

$$\langle\varphi^2\rangle^{(\text{int})}_{\text{b}} \approx -\frac{k_D(\xi-\xi_D)}{(4\pi)^{(D+1)/2}}\frac{D\Gamma((D-1)/2)}{(D-2)(a-y)^{D-2}}. \tag{40}$$

In the limit $am \gg 1$, the main contribution into the integral in formula (38) comes from the lower limit and one finds

$$\langle\varphi^2\rangle^{(\text{int})}_{\text{b}} \approx -\frac{Bm^{D-1}\cosh^2(my)}{(4\pi am)^{D/2}}\mathrm{e}^{-2am}, \qquad B \equiv \frac{D/2-2\xi D+m/k_D-\nu}{D/2-2\xi D-m/k_D-\nu}. \tag{41}$$

As we could expect, in this limit the VEVs are exponentially suppressed.

As in the case of the field square, the VEV for the EMT is presented in the form

$$\langle T_i^k\rangle^{(\text{int})}_{\text{ren}} = \langle T_i^k\rangle^{(\text{int})}_{0,\text{ren}} + \langle T_i^k\rangle^{(\text{int})}_{\text{b}}, \tag{42}$$

where $\langle T_{ik}\rangle^{(\text{int})}_{0,\text{ren}}$ is the vacuum EMT in the Minkowski spacetime orbifolded along the y-direction and the presence of the part $\langle T_{ik}\rangle^{(\text{int})}_{\text{b}}$ is related to that the geometry in the region $y > a$ is AdS. For the first part one has

$$\langle T_i^k\rangle^{(\text{int})}_{0,\text{ren}} = \frac{m^{D+1}\delta_i^k}{(4\pi my)^{\frac{D+1}{2}}}\left[K_{\frac{D+1}{2}}(2my)(2\xi-1)+(1-4\xi)myK_{\frac{D+3}{2}}(2my)\right], \tag{43}$$

with $i = 0, 1, \ldots, D-1$ and $\langle T_D^D\rangle^{(\text{int})}_{0,\text{ren}} = 0$. For the second term on the right of (42) we find (no summation over i)

$$\langle T_i^i\rangle^{(\text{int})}_{\text{b}} = \frac{(4\pi)^{-D/2}}{\Gamma(D/2)}\int_m^\infty \mathrm{d}x(x^2-m^2)^{D/2}U_\nu(x)$$
$$\times\left\{\frac{1}{D}\cosh^2(xy)+\frac{(4\xi-1)x^2}{x^2-m^2}\left[\cosh^2(xy)-\frac{1}{2}\right]\right\}, \tag{44}$$

$$\langle T_D^D\rangle^{(\text{int})}_{\text{b}} = -\frac{(4\pi)^{-D/2}}{2\Gamma(D/2)}\int_m^\infty \mathrm{d}x(x^2-m^2)^{D/2-1}x^2U_\nu(x), \tag{45}$$

with $i = 0, 1, \ldots, D-1$, and the function $U_\nu(x)$ is defined by formula (39). Note that the radial stress inside the brane does not depend on spacetime point. This result could also be obtained directly from the continuity equation. For a conformally coupled massless scalar we have $U_\nu(x) = 0$ and hence, the parts in the VEV of the EMT given by (44) and (45) vanish. In this case, the part $\langle T_i^i\rangle^{(\text{int})}_{0,\text{ren}}$ vanishes as well.

For the VEV of the EMT near the brane core we find (no summation over i)

$$\langle T_i^i \rangle_b^{(int)} \approx \frac{D-1}{Dk_D(y-a)} \langle T_D^D \rangle_b^{(int)} \approx \frac{Dk_D(\xi - \xi_D)^2}{2^D \pi^{(D+1)/2}} \frac{\Gamma((D+1)/2)}{(a-y)^D}, \tag{46}$$

with $i = 0, 1, \ldots, D-1$. For a conformally coupled scalar field the corresponding asymptotic behavior is given by formulae (32). In the limit $am \gg 1$ to the leading order one has (no summation over i)

$$\langle T_D^D \rangle_b^{(int)} \approx -\frac{Bm^{D+1} e^{-2am}}{2(4\pi am)^{D/2}}, \qquad \langle T_i^i \rangle_b^{(int)} \approx (1-4\xi)[2\cosh^2(my)-1]\langle T_D^D \rangle_b^{(int)}. \tag{47}$$

On the right panel of figure 2 we have plotted the dependence of the part in the VEV of the energy density induced by the exterior AdS geometry in the region inside the brane as a function of y/a for a minimally coupled massless scalar field in the case $D = 4$. The corresponding radial stress does not depend on y and $\langle T_D^D \rangle_b^{(int)} \approx 0.00134/a^5$ for $ak_D = 1$ and $\langle T_D^D \rangle_b^{(int)} \approx 0.000173/a^5$ for $ak_D = 0.1$. We recall that for a conformally coupled massless scalar field, the corresponding VEVs vanish for both field square and the EMT. The perpendicular interior vacuum force acting per unit surface of the brane boundary is determined by $-\langle T_D^D \rangle_b^{(int)}$. For minimally and conformally coupled scalars these forces tend to decrease the brane thickness.

5. Conclusion

We have considered the one-loop vacuum effects for a massive scalar field induced by a Z_2-symmetric thick brane on the $(D+1)$-dimensional AdS bulk. Among the most important characteristics of the vacuum, which carry information about the internal structure of the brane, are the VEVs for the field square and the EMT. In order to obtain these expectation values, we first construct the Wightman function. In the region outside the brane, this function is presented as a sum of two distinct contributions. The first one corresponds to the Wightman function in the free AdS geometry and the second one is induced by the brane. The latter is given by formula (13), where the tilted notation is defined by formula (14) with the coefficient from (15). This coefficient is determined by the radial part of the interior eigenfunctions and describes the influence of the core properties on the vacuum characteristics in the exterior region.

In section 3, we have investigated the influence of the non-trivial internal structure of the brane on the VEVs of the field square and the EMT. The parts in these VEVs induced by the brane are directly obtained from the corresponding part of the Wightman function. These parts are given by formulae (18) and (22) for the field square and the EMT respectively. For a conformally coupled massless scalar field the corresponding EMT vanishes. The parts in the VEVs of the field square and EMT induced by the brane diverge on the boundary of the brane. At large distances from the brane the brane-induced VEVs are suppressed by the factor e^{-2vk_Dy}. In the limit of the strong gravitational fields corresponding to large values of the AdS energy scale k_D, for points not too close to the brane the parts in the VEVs induced by the brane behave as $k_D^{D+1} e^{-2vk_D(y-a)}$ with upper/lower sign corresponding to the EMT/field square. In this case the relative contribution of the brane-induced effects is exponentially suppressed with respect to the free AdS part.

As an application of the general results, in section 4 we have considered a simple model with flat spacetime in the region inside the brane. The brane-induced parts of the exterior VEVs in this model are obtained from the general results by taking the function in the coefficient of the tilted notation from equation (28). We have also investigated the vacuum densities

inside the brane. Though the spacetime geometry inside the brane is Monkowskian, the AdS geometry of the exterior region induces vacuum polarization effects in this region as well. In order to find the corresponding renormalized VEVs of the field square and the EMT we have presented the Wightman function in the interior region in decomposed form (33). In this representation the first term on the right is the Wightman function in the Minkowski spacetime orbifolded along the direction perpendicular to the brane and the second one is induced by the AdS geometry in the exterior region. The corresponding parts in the VEVs of the field square and the EMT are given by formulae (38), (44) and (45). For a massless conformally coupled scalar field these parts vanish. In the general case of the curvature coupling parameter, the corresponding radial stress is uniform inside the brane and determines the interior vacuum forces acting on the boundary of the brane. For both minimally and conformally coupled scalar fields these forces tend to decrease the thickness of the brane. When the brane thickness tends to zero, from the formulae of the model with flat interior the corresponding results in the RS 1-brane model are obtained.

Acknowledgments

References

[1] Fabinger M and Horava P 2000 *Nucl. Phys.* B **580** 243
[2] Nojiri S, Odintsov S D and Zerbini S 2000 *Phys. Rev.* D **62** 064006
　　Nojiri S and Odintsov S 2000 *Phys. Lett.* B **484** 119
[3] Toms D J 2000 *Phys. Lett.* B **484** 149
[4] Nojiri S, Obregon O and Odintsov S D 2000 *Phys. Rev.* D **62** 104003
[5] Goldberger W and Rothstein I 2000 *Phys. Lett.* B **491** 339
[6] Nojiri S, Odintsov S D and Zerbini S 2000 *Class. Quantum Gravity* **17** 4855
　　Nojiri S and Odintsov S 2000 *J. High Energy Phys.* JHEP07(2000)049
[7] Garriga J, Pujolas O and Tanaka T 2001 *Nucl. Phys.* B **605** 192
[8] Mukohyama S 2000 *Phys. Rev.* D **63** 044008
[9] Hofmann R, Kanti P and Pospelov M 2001 *Phys. Rev.* D **63** 124020
[10] Brevik I, Milton K A, Nojiri S and Odintsov S D 2001 *Nucl. Phys.* B **599** 305
[11] Flachi A and Toms D J 2001 *Nucl. Phys.* B **610** 144
[12] Gilkey P B, Kirsten K and Vassilevich D V 2001 *Nucl. Phys.* B **601** 125
[13] Flachi A, Moss I G and Toms D J 2001 *Phys. Lett.* B **518** 153
　　Flachi A, Moss I G and Toms D J 2001 *Phys. Rev.* D **64** 105029
[14] Naylor W and Sasaki M 2002 *Phys. Lett.* B **542** 289
[15] Saharian A A and Setare M R 2003 *Phys. Lett.* B **552** 119
[16] Elizalde E, Nojiri S, Odintsov S D and Ogushi S 2003 *Phys. Rev.* D **67** 063515
[17] Garriga J and Pomarol A 2003 *Phys. Lett.* B **560** 91
[18] Nojiri S and Odintsov S D 2003 *J. Cosmol. Astropart. Phys.* JCAP06(2003)004
[19] Yeranyan A H and Saharian A A 2003 *Astrophysics* **46** 386
[20] Moss I G, Naylor W, Santiago-Germán W and Sasaki M 2003 *Phys. Rev.* D **67** 125010
[21] Saharian A A and Setare M R 2004 *Phys. Lett.* B **584** 306
[22] Cognola C, Elizalde E, Nojiri S, Odintsov S D and Zerbini S 2004 *Mod. Phys. Lett.* A **19** 1435
[23] Knapman A and Toms D J 2004 *Phys. Rev.* D **69** 044023
[24] Saharian A A 2004 *Astrophysics* **47** 303
[25] Nojiri S and Odintsov S D 2004 *Phys. Rev.* D **69** 023511
[26] Norman J P 2004 *Phys. Rev.* D **69** 125015
[27] Saharian A A 2004 *Phys. Rev.* D **70** 064026
　　Saharian A A 2005 *Astrophysics* **48** 122
[28] Pujolàs O and Tanaka T 2004 *J. Cosmol. Astropart. Phys.* JCAP12(2004)009

[29] Setare M R 2004 *Eur. Phys. J.* C **38** 373
[30] Saharian A A 2005 *Nucl. Phys.* B **712** 196
[31] Naylor W and Sasaki M 2005 *Prog. Theor. Phys.* **113** 535
[32] Saharian A A and Setare M R 2005 *Nucl. Phys.* B **724** 406
     Saharian A A and Setare M R 2007 *J. High Energy Phys.* JHEP02(2007)089
[33] Pujolàs O and Sasaki M 2005 *J. Cosmol. Astropart. Phys.* JCAP09(2005)002
[34] Setare M R 2005 *Phys. Lett.* B **620** 111
[35] Elizalde E 2006 *J. Phys.: Condens. Matter* A **39** 6299
[36] Minamitsuji M, Naylor W and Sasaki M 2006 *Nucl. Phys.* B **737** 121
     Minamitsuji M, Naylor W and Sasaki M 2006 *Phys. Lett.* B **633** 607
[37] Setare M R 2006 *Phys. Lett.* B **637** 1
[38] Saharian A A and Setare M R 2006 *Phys. Lett.* B **637** 5
[39] Elizalde E, Minamitsuji M and Naylor W 2007 Casimir effect in rugby-ball type flux compactifications *Preprint*
     hep-th/0702098
[40] Saharian A A and Mkhitaryan A L 2007 *J. High Energy Phys.* JHEP08(2007)063
[41] Minamitsuji M 2007 Casimir effect in a 6D warped flux compactification model *Preprint* 0704.3623
[42] Flachi A, Garriga J, Pujolàs O and Tanaka T 2003 *J. High Energy Phys.* JHEP08(2003)053
     Flachi A and Pujolàs O 2003 *Phys. Rev.* D **68** 025023
     Saharian A A 2006 *Phys. Rev.* D **73** 044012
     Saharian A A 2006 *Phys. Rev.* D **73** 064019
     Saharian A A 2006 *Phys. Rev.* D **74** 124009
[43] Randall L and Sundrum S 1998 *Phys. Rev. Lett.* **83** 4690
[44] Birrell N D and Davis P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
[45] Burgess C P and Lütken C A 1985 *Phys. Lett.* B **153** 137
     Camporesi R 1991 *Phys. Rev.* D **43** 3958
     Camela M and Burgess C P 1999 *Can. J. Phys.* **77** 85
     Caldarelli M M 1999 *Nucl. Phys.* B **549** 499
[46] Saharian A A 2004 *Phys. Rev.* D **69** 085005
[47] Prudnikov A P, Brychkov Yu A and Marichev O I 1986 *Integrals and Series* vol 2 (New York: Gordon and
     Breach)
[48] Allen B and Ottewill A C 1990 *Phys. Rev.* D **42** 2669
     Allen B, Kay B S and Ottewill A C 1996 *Phys. Rev.* D **53** 6829
     Bezerra de Mello E R, Bezerra V B, Saharian A A and Tarloyan A S 2006 *Phys. Rev.* D **74** 025017
     Bezerra de Mello E R and Saharian A A 2006 *J. High Energy Phys.* JHEP10(2006)049
     Bezerra de Mello E R and Saharian A A 2007 *Phys. Rev.* D **75** 065019